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LETTER TO THE EDITOR

Equivalent classes of critical circles

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Abstract. We present numerical evidence that the fractal properties of the critical invariant circles of a typical area-preserving twist map, as summarized by the $f(\alpha)$ spectrum and the generalized dimensions $D(q)$, depend only on the tails in the continued fraction expansion of the corresponding rotation numbers. $f(\alpha)$ and $D(q)$ are numerically the same for all critical invariant circles of the standard and sine maps which have the rotation numbers with the same periodic tail.

A typical Hamiltonian system possesses different types of regular orbits, such as periodic, quasiperiodic and homoclinic. In the case of area-preserving twist maps of a cylinder, such as the standard map (SM), given by the following equations,

$$T: \begin{cases} p_{t+1} = p_t - (k/2\pi)\sin(2\pi q_t) \\ q_{t+1} = q_t + p_{t+1} \qquad q_t \in \boldsymbol{S}^1,\ p_t \in \boldsymbol{R} \end{cases} \tag{1}$$

the set of periodic and quasiperiodic orbits is parametrized by the rotation number ν defined as follows,

$$\nu := \lim_{i\to\infty} \frac{\bar{\mathsf{T}}^i \bar{q} - \bar{q}}{i} \qquad \bar{q} \in \boldsymbol{R} \tag{2}$$

where $\bar{\mathsf{T}}$ is the lift of the map (1).

The closure of a typical quasiperiodic orbit for sufficiently small values of the perturbation parameter k is an analytic invariant circle. For a large value of the parameter the closure of the quasiperiodic orbit is a more complicated invariant set, called cantorus. It is an invariant Cantor set imbedded in the phase space.

At the critical value of the parameter, which depends on the rotation number and is denoted by $K(\nu)$, the quasiperiodic orbit is still dense on the invariant set which is at least homeomorphic to the circle. However, the invariant measure given by the orbit, that is the density of the orbit points, is singular with the respect to the Lebesque measure on the circle. The function which describes the density of the points of the critical quasiperiodic orbit has an intricate pattern of singularities. Thus, although the critical invariant set is homeomorphic to the circle, the critical quasiperiodic orbit is described by a fractal density function with a non-trivial self-similar structure. It is the purpose of this letter to analyse the dependence of the fractal properties of the critical orbits on their rotation numbers. The global self-similar structure of fractals with the non-trivial scaling is usually described by the spectrum of singularities $f(\alpha)$, related to the spectrum of generalized (Reneyi) dimensions [1].

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It is generally believed that the quasiperiodic orbits on the critical invariant circles, for a large class of area-preserving maps, are all qualitatively similar. Results of the method of modular smoothing [2] show that the dominant singularities of the critical circles are of the same type for circles with the rotation numbers related by a modular transformation. Such rotation numbers have the same tail in their continued fraction expansion (CFE). The idea that the qualitative behaviour of the critical orbits should depend only on the tail of the CFE of the corresponding rotation number is also central in renormalization theory for arbitrary rotation numbers [3].

One way to describe these observations is by considering fractal properties of the invariant measure on the circle given by the critical quasiperiodic orbits. Our numerical calculations, presented in this letter, indicate that the singularity spectrum $f(\alpha)$ and the spectrum of fractal dimensions $D(q)$ of the invariant measure $\mu(\nu)$ are the same for all orbits of the SM with the rotation numbers with the same tails in the CFE, and are different if the tails of the CFE are different. Thus, the critical circles of the SM can be divided into equivalent classes with respect to their fractal properties. Members of the same class have the same $f(\alpha)$ and $D(q)$ and the tail in the CFE and different tails imply different $f(\alpha)$ and $D(q)$.

The fractal properties of the critical tori were described by $f(\alpha)$ and $D(q)$ for the first time in [4]. However, Osbaldestin and Sarkis calculated $f(\alpha)$ and $D(q)$ for a few critical circles without systematic exploration and discussion of the dependence on the number-theoretical properties of the rotation numbers. The result that the information dimension D_1 is the same for all orbits with the same tail in the CFE of their rotation numbers is implicitly contained in the recent work of Hunt *et al* [5]. Our results show that not only D_1 but the whole spectrum $D(q)$ depends only on the tail of the rotation number.

As pointed out before, the critical orbit can still be considered as an orbit of a homeomorphism of the circle. To define the function $f(\alpha)$ one considers an infinite set of partitions of the circle. The Nth partition contains N pieces labelled by an index $i : 1 \leqslant i \leqslant N$. The size of the ith piece is denoted by l_i, and the probability that a point of the orbit is in the ith piece is denoted by p_i. One then assumes that p_i scales as $p_i \approx l_i^{\alpha}$, and defines the function $f(\alpha)$ as the Hausdorf dimension of the set of points having exponent α. If the partitions and the measure are appropriate for the considered self-similar structure then the function $f(\alpha)$ can be calculated from the properties of the limit of the sequence of partitions.

To compute $f(\alpha)$ of the critical circle with an arbitrary rotation number ν we follow the procedure used in [6] for computation of $f(\alpha)$ for the critical circle with the rotation number equal to the golden mean $\nu = [0, 1^{\infty}]$. Similar, but not the same, partitions which gave the same asymptotic results were used in [4, 5]. In our computations the partitions of the circle are given by the points of the periodic orbits which approximate the critical quasiperiodic orbit. The rotation numbers of the set of periodic orbits are chosen as the successive continued fraction approximants of the irrational $\nu = [0, a_1, a_2 \ldots]$. Thus the ith partition of the circle is given by the n points of the periodic orbit with the rotation number $m/n = [0, a_1, a_2, \ldots, a_i]$ at the parameter value $k = K(\nu)$. There are two such orbits, one elliptic and one hyperbolic, but our results for $f(\alpha)$ and $D(q)$ did not depend on which of the two types of orbits were used to generate the partitions. If (p_i, q_i) and (p_{i+1}, q_{i+1}) are coordinates of the two neighbouring points on the periodic orbit then

$$l_i(n) = [(p_i - p_{i+1})^2 + (q_i - q_{i+1})^2]^{1/2}. \tag{3}$$

The measure of $l_i(n)$ is defined as $p_i(n) = 1/n$, and the partition function of the partition with n points is then given by

$$\Gamma_n(q_n, \tau_n) = \sum_{i=1}^{i=n} \frac{p_i^{q_n}}{l_i^{\tau_n}} = n^{-q_n} \sum_{i=1}^{i=n} l_i^{\tau_n}. \tag{4}$$

If the partitions and measures are appropriate for the considered fractal then the partition function is of the order of unity when

$$\tau_n = (q_n - 1) D_n(q_n) \tag{5}$$

and

$$D(q) = \lim_{n\to\infty} D_n(q_n) \tag{6}$$

where $D(q)$ is the set of generalized (Reneyi) dimensions of the fractal. $f(\alpha)$ is then given by

$$f_n(\alpha_n) = q_n\alpha_n(\tau_n) - \tau_n \qquad \alpha_n(\tau_n) = \frac{\mathrm{d}\tau_n}{\mathrm{d}q_n} \tag{7}$$

and

$$f(\alpha) = \lim_{n\to\infty} f_n(\alpha_n). \tag{8}$$

If the results are convergent than the choice of measure and the partitions are indeed appropriate for the considered asymptotically self-similar structure. In summary, the procedure to calculate $f(\alpha)$ and $D(q)$ for a critical quasiperiodic orbit consists of the following steps. Estimate the critical value $K(\nu)$, using for example the Greene criterion [7], and then calculate $\tau_n(q_n)$ for the partitions generated by the periodic orbits with the rotation numbers given by the successive continued fraction convergents. Each $\tau_n(q_n)$ gives the corresponding α_n, $f_n(\alpha_n)$ and $D_n(q_n)$. Finally, estimate the limit of $f_n(\alpha_n)$ and $D_n(q_n)$. The convergence can be enhanced by the usual ratio trick, and the condition that $\Gamma(\tau_n, q_n)/\Gamma(\tau_{n+1}, q_{n+1}) = 1$ can be used to provide explicit formulae for q_n and α_n as functions of τ_n [6].

Our main result is illustrated in figures 1(a) and (b). The figures represent $f(\alpha)$ and $D(q)$ for the critical quasiperiodic orbits with specially selected rotation numbers. The full curve illustrates $f(\alpha)$ and $D(q)$ for a class of quasiperiodic orbits with the same tail in the CFE as the tail of the golden mean, that is $[1^\infty]$. The curves for quasiperiodic orbits with different rotation numbers in this class are indistinguishable. The same result is true for the classes given by other periodic tails. The dashed curve represents $f(\alpha)$ and $D(q)$ for the class with the tail equal to $[2^\infty]$, the dotted curve represents the class with the tail equal to $[3^\infty]$ and the dot-dashed curve represents the class with the tail $[4^\infty]$. The functions $f(\alpha)$ and $D(q)$ for different classes are different but within the class, given by the tail of the CFE of the rotation number, these functions are the same. In the calculations we considered only rotation numbers with periodic tail and various initial segments of the CFE. The reason for such limitation is numerical. These are the orbits which are most easily approximated numerically by the periodic convergents, and the estimates of $f(\alpha)$ and $D(q)$ are the most reliable for such orbits. However, we cannot forsee any deeper reason that could make our conclusions invalid for other irrationals, which satisfy the general conditions of the Kolmogorov-Arnol'd-Moser (KAM) theory.

Our calculations of $f(\alpha)$ and $D(q)$ require knowledge of quite long periodic orbits at the critical values of the parameter. These orbits are used to determine the values of $K(\nu)$ and to calculate the partition functions (4). The final results, that is $f(\alpha)$ and $D(q)$, are extremely sensitive to the value of the parameter k. In order to establish our conclusions

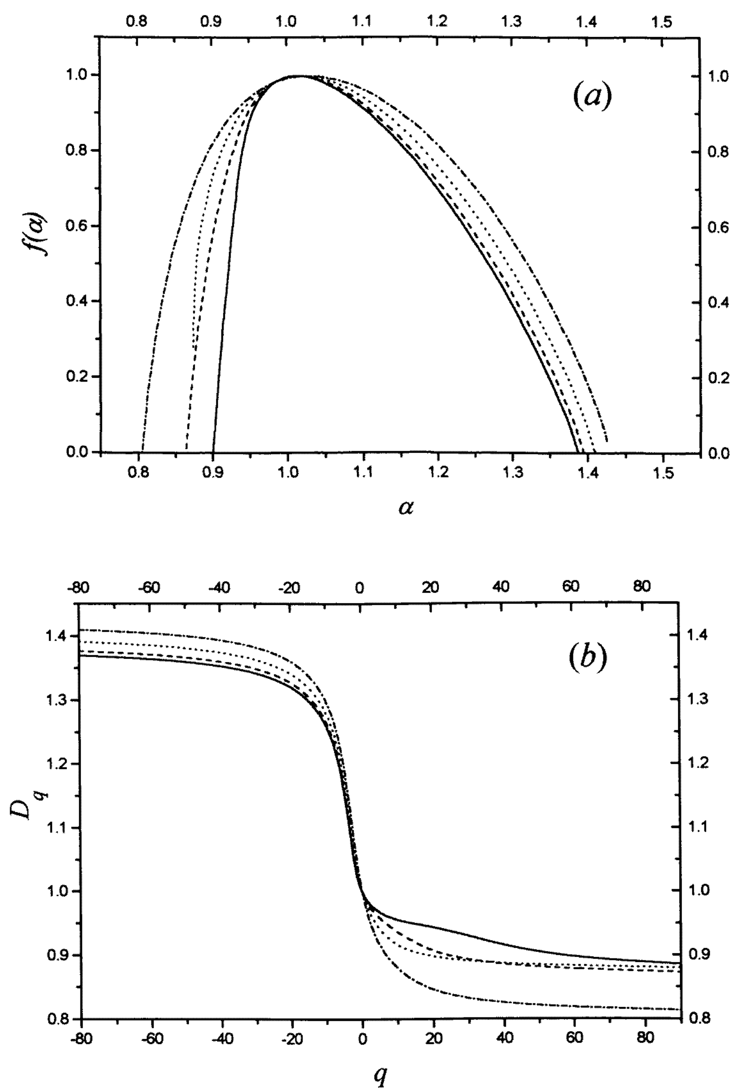

Figure 1. $f(\alpha)$ (a) and $D(q)$ (b) curves for the critical circles in several equivalence classes. Shown are the classes corresponding to the tails equal to $[1^\infty]$ (full), $[2^\infty]$ (dashed), $[3^\infty]$ (dotted) and $[4^\infty]$ (dot-dashed).

we need to calculate $K(\nu)$ with at least five significant digits, which implies calculations of very long periodic orbits. If one of the initial coefficients in the CFE of the rotation number is large than the long periodic orbit is in a small neighbourhood of a very unstable periodic orbit. Calculation of such orbits is obviously a very difficult numerical problem. This has limited us to a relatively small set of about five quasiperiodic orbits in each considered class, which, we believe, is still typical and large enough to put confidence in our conclusions.

Finally, we would like to comment on the more frequently discussed [5, 6] type of universality of the fractal properties of the critical circles. It is the universality of the fractal properties of the critical circles with the same rotation number but for different area-preserving maps in a certain class. We have calculated $f(\alpha)$ and $D(q)$ for the sine map obtained from the SM by replacing $\sin(2\pi q)$ by $\frac{1}{2}\sin(2\pi q) + \frac{1}{4}\sin(4\pi q)$, and obtained the same results as for the standard map for all calculated critical circles. This extends results of [5] where only the information dimension D_1 was considered.

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